1. The curve C has equation

$$y = 2x^2 - 12x + 16$$

Find the gradient of the curve at the point P (5, 6).

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(4)

$y = 2x^2 - 12x + 16$

$\frac{dy}{dx} = \underline{4x - 12}$

when $x = 5$, $\frac{dy}{dx} = 4(5) - 12$

$= 20 - 12$

$\boxed{= 8}$

$\therefore$ the gradient at $P(5,6)$ is 8

2. A curve has equation

$$y = 3x^2 + \frac{24}{x} + 2 \qquad x > 0$$

(a) Find, in simplest form, $\frac{dy}{dx}$

(3)

(b) Hence find the exact range of values of x for which the curve is increasing.

(2)

a) $y = 3x^2 + 24x^{-1} + 2$

$\frac{dy}{dx} = 6x - 24x^{-2} = \boxed{6x - \frac{24}{x^2}}$

b) curve is increasing; $\frac{dy}{dx} > 0$

$\Rightarrow \quad 6x - \frac{24}{x^2} > 0$

$\Rightarrow \quad 6x^3 - 24 > 0$

$\Rightarrow \quad x^3 > 4$

$\Rightarrow \quad \boxed{x > 4^{\frac{1}{3}}}$

3.

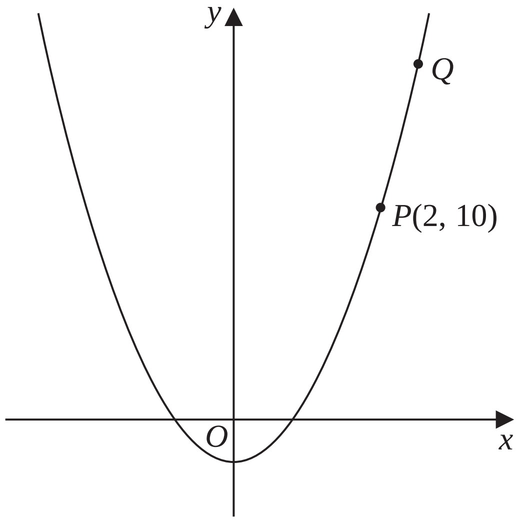


Figure 1

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P.

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ, giving your answer in terms of h in simplest form.

(3)

(c) Explain briefly the relationship between part (b) and the answer to part (a).

(1)

(a) $y = 3x^2 - 2$

$\frac{dy}{dx} = 6x$ (1)

gradient at $P(2,10) = 6(2) = 12$ (1)

(b) First we find coordinates of Q in terms of h

x coordinate is given : $2+h$

Find y coordinate : $y = 3x^2 - 2$

$y = 3(2+h)^2 - 2$

$= 3(4 + 4h + h^2) - 2$

$= 12 + 12h + 3h^2 - 2$

$= 3h^2 + 12h + 10$

Find gradient of PQ $= \dfrac{y_1 - y_2}{x_1 - x_2}$

$= \dfrac{(3h^2 + 12h + 10) - 10}{(2+h) - 2}$ 1

$= \dfrac{3h^2 + 12h}{h}$ 1

$= 3h + 12$ * 1

(c) $h \to 0$, $3h + 12 \to 12$

$\therefore$ The gradient of the chord tends to the gradient of the tangent to the curve. * 1

4. The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point (2, 10) lies on C
- the gradient of the curve at (2, 10) is -3

(a) (i) show that the value of a is -2

(ii) find the value of b.

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write $f(x)$ in the form $(x - 4)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)

(a)(i) $f(x) = ax^3 + 15x^2 - 39x + b$

$f'(x) = 3ax^2 + 30x - 39$

$f'(2) = 3a(2)^2 + 30(2) - 39$

$= 12a + 60 - 39$

$f'(2) = 12a + 21$

Since we were given gradient is -3, substitute this with $f'(2)$.

$f'(2) = 12a + 21$

$-3 = 12a + 21$ (1)

$-24 = 12a$

$a = -2$ (1)

(a)(ii) $f(x) = ax^3 + 15x^2 - 39x + b$

$= -2x^3 + 15x^2 - 39x + b$

$10 = -2(2)^3 + 15(2)^2 - 39(2) + b$ (1)

$10 = -16 + 60 - 78 + b$

$10 = -34 + b$

$b = 44$ (1)

(b) $f(x) = -2x^3 + 15x^2 - 39x + 44$

$\frac{dy}{dx} = -6x^2 + 30x - 39$ (1)

$b^2 - 4ac = (30)^2 - 4(-6)(-39)$

$= -36$ (1)

As $b^2 - 4ac = -36 < 0$, $f'(x) \neq 0$. This means that $f'(x)$ has no real roots. Hence, $f'(x)$ has no turning points. (1)

(c) $-2x^3 + 15x^2 - 39x + 44 \equiv (x-4)(ax^2 + bx + c)$

$\equiv ax^3 + bx^2 + cx - 4ax^2 - 4bx - 4c$

$\equiv ax^3 + bx^2 - 4ax^2 + cx - 4bx - 4c$

$\equiv ax^3 + (b-4a)x^2 + (c-4b)x - 4c$

(1) $a = -2$ (2) $b - 4a = 15$ (3) $-4c = 44$ (1)

$b - 4(-2) = 15$ $c = -11$

$b + 8 = 15$

$b = 7$

Hence, $f(x) = (x-4)(-2x^2 + 7x - 11)$ (1)

(d) The curve intersects the y-axis when $x = 0$. So, substitute $x = 0$ into the equation :

$$f(0) = (0-4)(-2(0)^2 + 7(0) - 11)$$
$$= -4(-11)$$
$$= 44$$

Points of intersection with y-axis $= (0, 44)$

The curve intersects the x-axis when $y = 0$. So :

$f(x) = 0 \rightarrow (x-4)(-2x^2 + 7x - 11) = 0$ solving this using calculator will not give us real roots

$\therefore x = 4$ is the only real solution

$f(x) = 0 \rightarrow (4, 0)$

The question asks for $f(0.2x)$. $f(x)$ to $f(\frac{1}{5}x)$ has a scalar factor of 5 in the x-direction. So, $(4,0)$ needs to be multiplied by 5

$(4, 0) \times 5 = (20, 0)$

$\therefore$ Hence, the points of intersections are $(0, 44)$ ① and $(20, 0)$ ① #

where $f(0.2x)$ intersects y-axis

where $f(0.2x)$ intersects x-axis

5. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \qquad x > 0$$

(a) Find (i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$

(3)

(b) Verify that C has a stationary point when $x = 4$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

a) i) $y = x^2 - 2x - 24x^{1/2}$ $\qquad$ $y = ax^n, \dfrac{dy}{dx} = anx^{n-1}$

$\dfrac{dy}{dx} = 2x - 2 - 12x^{-1/2}$ ✓✓

ii) $\dfrac{d^2y}{dx^2} = 2 + 6x^{-3/2}$ ✓ $\qquad$ $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

b) $\dfrac{dy}{dx} = 2x - 2 - 12x^{-1/2}$

S.p. is when $\left.\dfrac{dy}{dx}\right|_{x=a} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=4} = 2(4) - 2 - 12(4)^{-1/2}$ ✓

$= 8 - 2 - 6 = 0$ ✓

$\left.\dfrac{dy}{dx}\right|_{x=4} = 0$ ✓, hence a stationary point at $x = 4$. ✓

c) $\frac{d^2y}{dx^2} = 2 + 6x^{-3/2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=a} > 0 \rightarrow$ minimum

$\left.\frac{d^2y}{dx^2}\right|_{x=b} < 0 \rightarrow$ maximum

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = 2 + 6(4)^{-3/2}$$

$$= 2 + \frac{6}{8} = 2.75$$

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = 2.75 > 0$, hence stationary point is a minimum

6. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

a) $x = 4\sin 2y$

$\frac{dx}{dy} = 4(2\cos 2y)$ ①

$= 8\cos 2y$

Take reciprocal

$\frac{dy}{dx} = \frac{1}{8\cos 2y}$

At origin $(0,0)$ so sub $y=0$

$\frac{dy}{dx} = \frac{1}{8\cos(0)}$

$\cos(0) = 1$

$\frac{dy}{dx} = \frac{1}{8}$ ①

bi) $\sin x \approx x$

small angle approximation

$\sin 2y \approx 2y$ ①

$\therefore x = 4\sin 2y$

using $\sin 2y \approx 2y$

$x \approx 4(2y)$

$x \approx 8y$

bii) Value found in a) is the gradient of the line found in bi) ①

$y = \frac{1}{8}x$

can see by re-arranging that gradient same as value in a)

c) $\frac{dy}{dx} = \frac{1}{8\cos 2y}$

$\sin^2 x + \cos^2 x = 1$

$\therefore \sin^2 2y + \cos^2 2y = 1$

$x = 4\sin 2y$

$x^2 = 16\sin^2 2y$

using $\sin^2 2y = 1 - \cos^2 2y$ (1)

$x^2 = 16(1 - \cos^2 2y)$

$x^2 = 16 - 16\cos^2 2y$ (1)

$16\cos^2 2y = 16 - x^2$

$\cos^2 2y = 1 - \frac{x^2}{16}$

$\cos 2y = \sqrt{1 - \frac{x^2}{16}}$

$\frac{dy}{dx} = \frac{1}{8\sqrt{1 - \frac{x^2}{16}}} \begin{matrix} \times\sqrt{16} \\ \times\sqrt{16} \end{matrix}$

$= \frac{\sqrt{\cancel{16}}\ 4}{8\sqrt{16 - x^2}}$

$= \frac{1}{2\sqrt{16 - x^2}}$ (1)

7.

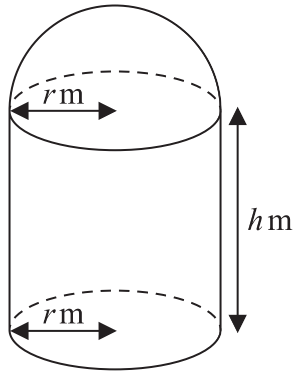


Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is $6\,\text{m}^3$.

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)

a) $A = A_1 + A_2 + A_3$

Area Cylinder : $A_1 = 2\pi rh$

Area base : $A_2 = \pi r^2$

Area hemisphere : $A_3 = 2\pi r^2$

$V = 6\text{m}^3 = V_{cylinder} + V_{hsphere}$

$6 = \pi r^2 h + \frac{2\pi r^3}{3}$ ① $\Rightarrow \pi r^2\left(h + \frac{2}{3}r\right) = 6$

$\Rightarrow h = \frac{6}{\pi r^2} - \frac{2}{3}r$ ①

$\Rightarrow A = 2\pi rh + \pi r^2 + 2\pi r^2$ ①

$\Rightarrow A = 2\pi rh + 3\pi r^2$

$\Rightarrow A = \left(\frac{6}{\pi r^2} - \frac{2}{3}r\right)2\pi r + 3\pi r^2$

$\Rightarrow A = \frac{12\pi r}{\pi r^2} - \frac{4}{3}\pi r^2 + 3\pi r^2$

$\Rightarrow A = \frac{12}{r} + \frac{5}{3}\pi r^2$ as required. ①

b) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$

1 Differentiate
2 Set equal 0
3 Solve for r

$\frac{dA}{dr} = -\frac{12}{r^2} + \frac{10\pi r}{3} = 0$ 2

$\Rightarrow -\frac{36}{r^2} + 10\pi r = 0$

$\Rightarrow -36 + 10\pi r^3 = 0 \Rightarrow 10\pi r^3 = 36$

$\Rightarrow r^3 = \frac{36}{10\pi}$ 1 $\Rightarrow r = \sqrt[3]{\frac{36}{10\pi}} = 1.05m$

$\Rightarrow$ Surface area will be a minimum when $r = 1.05m$ 1

c) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$, $r = 1.05m$

$A = \frac{12}{1.05} + \frac{5}{3}\pi(1.05)^2 = 17.201...$ 1

$\Rightarrow$ Minimum Surface is $17m^2$ 1

8. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

a)

i) $y = 3x^4 - 8x^3 - 3 \Rightarrow \frac{dy}{dx} = \underline{\underline{12x^3 - 24x^2}}$ 1 1

ii) $\frac{d^2y}{dx^2} = \underline{\underline{36x^2 - 48x}}$ 1

b) Stationary point when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 12x^3 - 24x^2 \Rightarrow 12(2)^3 - 24(2)^2 = 12 \times 8 - 24 \times 4 = 0$ 1

$\Rightarrow$ At $x = 2$, $\frac{dy}{dx} = 0 \Rightarrow x = 2$ is a stationary point. 1

c) $\frac{d^2y}{dx^2}$ and substitute in $x = 2$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ Minimum

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ Maximum

$\left.\frac{d^2y}{dx^2}\right|_2 = 36(2)^2 - 48(2) = 144 - 96 = 48$ 1

$\Rightarrow 48 > 0 \Rightarrow$ Stationary point which is a $\underline{\underline{\text{minimum}}}$. 1

9. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) i) $\dfrac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ (1) (1)

use rule $ax^n \rightarrow anx^{n-1}$ to differentiate x

e.g. $5x^4 \rightarrow 5 \times 4 \times x^{4-1} = 20x^3$

ii) $\dfrac{d^2y}{dx^2} = 60x^2 - 144x + 84$ (1)

b) i) when $x = 1$, gradient $\dfrac{dy}{dx} = 20(1^3) - 72(1^2) + 84(1) - 32$ (1)

$= 20 - 72 + 84 - 32$

$= 0$

$\therefore \dfrac{dy}{dx} = 0$ so there is a stationary point at $x = 1$. (1)

ii) when $x = 0.8$, $\dfrac{dy}{dx} = 20(0.8^3) - 72(0.8^2) + 84(0.8) - 32$ ← choose x values either side of the stationary point.

$= -0.64$

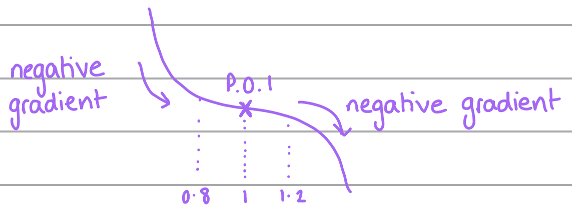
$-0.64 < 0$ so gradient is negative

when $x = 1.2$, $\dfrac{dy}{dx} = 20(1.2^3) - 72(1.2^2) + 84(1.2) - 32$

$= -0.32$

$-0.32 < 0$ so gradient is negative. (1) for finding both gradients

Since the gradient is negative on both sides of the stationary point, this must be a point of inflection. (1)



10. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

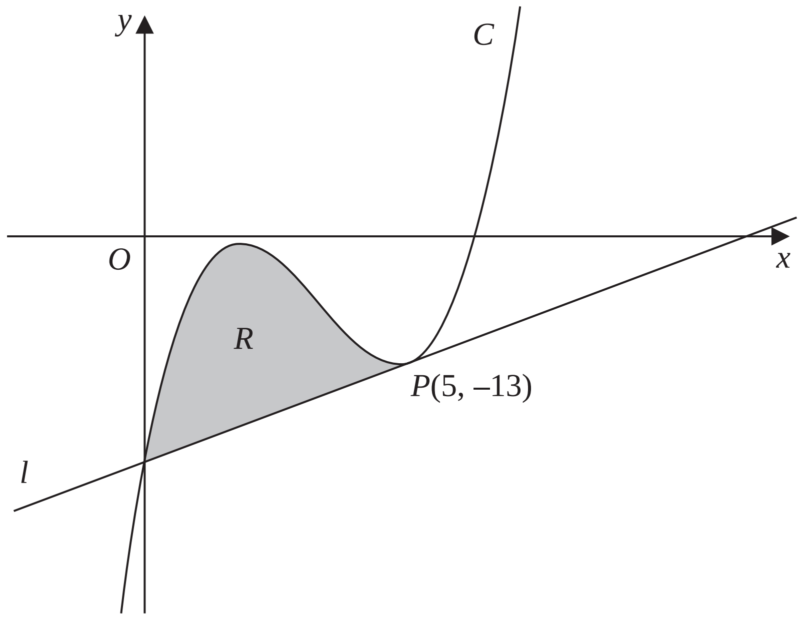


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

a) $y = x^3 - 10x^2 + 27x - 23$ — we know the point $(5, -13)$ is on the line l.

$\frac{dy}{dx} = 3x^2 - 20x + 27$ 1

when $x = 5$, gradient $\frac{dy}{dx} = 3(5^2) - 20(5) + 27 = 2$ 1

1 use formula $y - y_1 = m(x - x_1)$ with point $(5, -13)$

$y - (-13) = 2(x - 5)$

$y + 13 = 2(x - 5)$

$y = 2x - 23$ 1

Question continued

b) when $x = 0$ (on the y-axis).

L: $y = 2(0) - 23 = -23$

C: $y = 0^3 - 10(0^2) + 27(0) - 23 = -23$

Both C and L pass through $(0, -23)$, so C meets L again on the y-axis.

difference between area bound by C and area bound by L.

c) $R = \int_0^5 (x^3 - 10x^2 + 27x - 23 - (2x - 23))\,dx$

$R = \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{25x^2}{2}\right]_0^5$ 1 for correct integration 1 for applying correct bounds

$R = \frac{625}{4} - \frac{1250}{3} + \frac{625}{2}$ 1

$R = \frac{625}{12}$ 1

to integrate ax^n.

$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} (+c)$

eg $\int 27x\,dx = \frac{27}{1+1}x^{1+1}$

$= \frac{27}{2}x^2$